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# Flux-Vector Splitting for the 1990s

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## Abstract

The development of flux-vector splitting through the 1970s and 1980s is reviewed. Attention is given to the diffusive nature of flux-vector splitting, which makes it an undesirable technique for approximating the inviscid fluxes in a Navier-Stokes solver. Several proposed improvements, including a brand-new one, are discussed and illustrated by a simple, yet revealing, numerical test case. Finally, an outlook for flux-vector splitting in the 1990s is presented.

## 1 Introduction

Flux-vector splitting is a technique for achieving upwind bias in numerical flux functions for the Euler equations. It has been very popular during the Euler era (the 1980s), but has lost much of its appeal with the rise of Navier-Stokes solvers, since it causes undesirably high errors in boundary layers. Ironically, it shares this deficiency with other flux functions that are widely used in Navier-Stokes codes [1].

Flux-vector splitting is a natural consequence of regarding a fluid as an ensemble of particles. Measured along any coordinate, some particles will move forward, other ones backward; this automatically splits the fluxes of

mass, momentum and energy into forward and backward fluxes, i.e.

$$\mathbf{F} = \mathbf{F}^+ + \mathbf{F}^-. \quad (1)$$

On a computational grid these split fluxes immediately allow upwind differencing:

$$\left( \frac{\partial \mathbf{F}}{\partial x} \right)_j := \frac{\mathbf{F}_j^+ - \mathbf{F}_{j-1}^+}{\Delta x} + \frac{\mathbf{F}_{j+1}^- - \mathbf{F}_j^-}{\Delta x}. \quad (2)$$

A full update step with the first-order upwind flux-split scheme for the Euler equations can be regarded as an approximate way to integrate the collisionless Boltzmann equation

$$\frac{\partial \phi(q)}{\partial t} + q \frac{\partial \phi(q)}{\partial x} = 0 \quad (3)$$

for the distribution function  $\phi(q)$  of the particle velocity  $q$ . For the sake of numerical simplicity the velocity distribution in practice usually is not chosen to be Maxwellian (see, however, [2]). Because of this, and the finite length of the time-step, the velocity distribution in any cell at the end of a time-step is not of the same functional form as at the start of the time-step, when it was assumed to be an “equilibrium” distribution. Replacing the final, non-equilibrium distribution by an equilibrium distribution with the same integrated mass, momentum and energy in each cell instantaneously simulates the effect of the particle collisions. Euler schemes based on flux-vector splitting may therefore be termed Boltzmann-type schemes [3].

## 2 A gallery of flux-vector splittings

### 2.1 The Beam Scheme, alias Steger-Warming splitting

The first Boltzmann-type scheme in use for integrating the Euler equations was developed for astrophysical calculations around 1970 by Kevin Prendergast [4]; it is based on a velocity distribution consisting of delta-functions, whence the name “Beam Scheme”:

$$\phi(q) = \rho_s \delta[q - (u - \sigma)] + (\rho - 2\rho_s) \delta(q - u) + \rho_s \delta[q - (u + \sigma)]. \quad (4)$$

Here  $\rho$  is the density,  $\rho_s$  is the density of the particles in each side beam,  $u$  is the mass-averaged flow velocity, and  $\sigma$  is a dispersion velocity not less than the speed of sound. In the special case that  $\sigma$  equals the speed of sound, and the specific internal energy that is *not* in the form of  $x$ -translations (in the Beam Scheme a passively convected quantity) is equidistributed over the side beams, the flux splitting becomes identical to the well-known splitting of Steger and Warming [5], developed in the late 1970s without regard to a velocity distribution. This equivalence was discovered by Van Albada [6].

The Beam-Scheme interpretation is advantageous, for instance, when formulating the flux splitting for gases with an equation of state more complicated than the ideal-gas law [7].

## 2.2 Van Leer's splitting

The most popular splitting subsequently developed, Van Leer's [8], again is not derived from or associated with any *a priori* chosen velocity distribution, but from certain mathematical constraints. The main design features of Van Leer's splitting are:

- the split fluxes are continuously differentiable (Beam/Steger-Warming fluxes are non-differentiable in sonic and stagnation points, which is inappropriate when approximating a first-order system of conservation laws);
- for subsonic flow the Jacobians of the split fluxes have a zero eigenvalue, which accounts for crisp numerical profiles of stationary shocks.

If the second constraint is relaxed, a one-parameter family of continuously differentiable split fluxes can be generated [9]; these are the simplest possible in the sense that they are at most quartic in the Mach number, just as the Van Leer fluxes. The differences among members of this family arise only in the energy-flux splitting.

Extensions of these fluxes to real gases are also included in [9], but have been given by several other authors [7] [10].

### 2.3 Hänel's energy-flux splitting

The above family of energy-flux splittings includes the one originally proposed by Hänel [11]:

$$F_{\text{energy}}^{\pm} = F_{\text{mass}}^{\pm} H, \quad (5)$$

where  $H$  is the specific total enthalpy. Advantages of this formula are:

- it is as simple as can be;
- it admits steady Euler solutions with constant total enthalpy throughout the flow.

Hänel claims that this flux splitting, when used in Navier-Stokes calculations, gives more accurate total-enthalpy values in the boundary layer. This may have been observed for the lower flow speeds; in the hypersonic flow regime the improvement is insignificant (see further in Section 5).

## 3 Flux-vector splitting as an approximate Riemann solver

Flux-vector splitting can be used beyond the first-order upwind scheme; in a higher-order MUSCL-type code [12] it simply takes the same place as any “approximate Riemann solver,” such as Roe’s [13] or Osher’s [14]. It is one more way to merge the two state vectors on the left and right side of a cell interface into one flux vector, namely, by

$$\mathbf{F}(\mathbf{U}_L, \mathbf{U}_R) = \mathbf{F}^+(\mathbf{U}_L) + \mathbf{F}^-(\mathbf{U}_R). \quad (6)$$

## 4 Dissatisfaction with flux-vector splitting

Dissatisfaction with flux-vector splitting in Navier-Stokes codes stems from the dissipative properties of flux-vector splitting: contact discontinuities and slip surfaces are diffused, even when these are stationary and aligned with the grid [8] [3]. This is in contrast to flux-difference splittings such as Roe’s and Osher’s. A serious consequence is that the attached boundary layers to

be resolved by Navier-Stokes codes get artificially broadened, and adiabatic-wall temperatures become inaccurate; numerical solutions obtained with flux-difference splitting are vastly superior [1].

This can be easily understood from, for instance, the net transverse-momentum flux across the boundary layer, as computed with flux-vector splitting ( $B$  denotes bottom cell,  $T$  top cell):

$$\rho v u : = F_{\text{mass}B}^+ u_B + F_{\text{mass}T}^- u_T \quad (7)$$

$$= (F_{\text{mass}B}^+ + F_{\text{mass}T}^-) \frac{u_B + u_T}{2} - (F_{\text{mass}B}^+ - F_{\text{mass}T}^-) \frac{u_T - u_B}{2}. \quad (8)$$

When the net mass flux

$$F_{\text{mass}}^{\text{net}} = F_{\text{mass}B}^+ + F_{\text{mass}T}^- \quad (9)$$

vanishes, the second term in the momentum flux still causes numerical diffusion.

## 5 Improvements to flux formulas

Hänel [15] has suggested to replace the above formula for the transverse-momentum flux by one borrowed from *flux-difference* splitting:

$$\rho v u := F_{\text{mass}}^{\text{net}} u_{\text{upwind}}, \quad (10)$$

with

$$u_{\text{upwind}} = u_B \quad \text{if } F_{\text{mass}}^{\text{net}} \geq 0, \quad (11)$$

$$u_{\text{upwind}} = u_T \quad \text{if } F_{\text{mass}}^{\text{net}} < 0. \quad (12)$$

This mixture of flux-vector splitting and flux-difference splitting prevents the numerical broadening of the boundary layer, but does not improve the accuracy of the wall temperature. It further introduces pressure irregularities across the boundary layer.

Nevertheless, the partial success of Hänel's modified transverse-momentum flux suggests that a further improvement can be obtained by introducing a similar formula for the energy flux, i.e.:

$$\rho v H := F_{\text{mass}}^{\text{net}} H_{\text{upwind}}, \quad (13)$$

with

$$H_{\text{upwind}} = H_B \quad \text{if } F_{\text{mass}}^{\text{net}} \geq 0, \quad (14)$$

$$H_{\text{upwind}} = H_T \quad \text{if } F_{\text{mass}}^{\text{net}} < 0. \quad (15)$$

A numerical test confirms the improvement, as shown in Figures 1 and 2. Plotted are the temperature and pressure distributions from the solution to the conical-flow problem of [1], as computed with the Roe fluxes, the Van Leer fluxes, the two Hanel modifications of Equations 5 and 10, and the extra modification 13 suggested in this section. The wall temperature computed with Equation 13 is significantly closer to the correct value of 11.7 than for the other Van Leer-type splittings; unfortunately, the pressure irregularity remains. Convergence histories for the four calculations are shown in Figure 3 and indicate that the modified schemes are as robust as the original Van Leer scheme.

It is clear that we have hardly begun to investigate mixtures of flux-vector and flux-difference splitting; the success of the formula introduced in this paper suggests that there is something to gain here. The advantage of a simple blended formula over full flux-difference splitting is that it will allow a complete linearization for use in implicit schemes; for Roe's flux formula, for instance, only an incomplete linearization is acceptable in practice (see the horrendous complete formula in [16]).

## 6 Can pure flux-vector splitting be saved?

However promising the above improvement may be, an intriguing question still remains:

- Is it possible at all to construct a flux-vector splitting that does not diffuse a grid-aligned boundary layer?

This question can be rephrased as:

- Is it possible to split the Euler fluxes such that both  $F_{\text{mass}}^+$  and  $F_{\text{mass}}^-$  vanish with the flow speed?

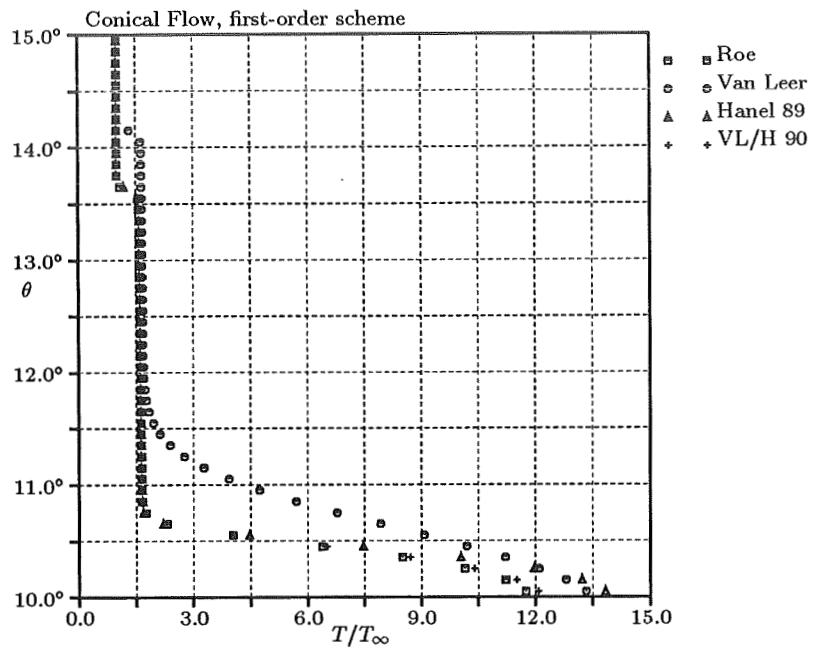


Figure 1: Numerical solution of the Navier-Stokes equations for self-similar hypersonic flow ( $M_\infty = 7.95$ ) over a  $10^\circ$  cone; shown are the non-dimensional temperature distributions computed with four different flux functions.

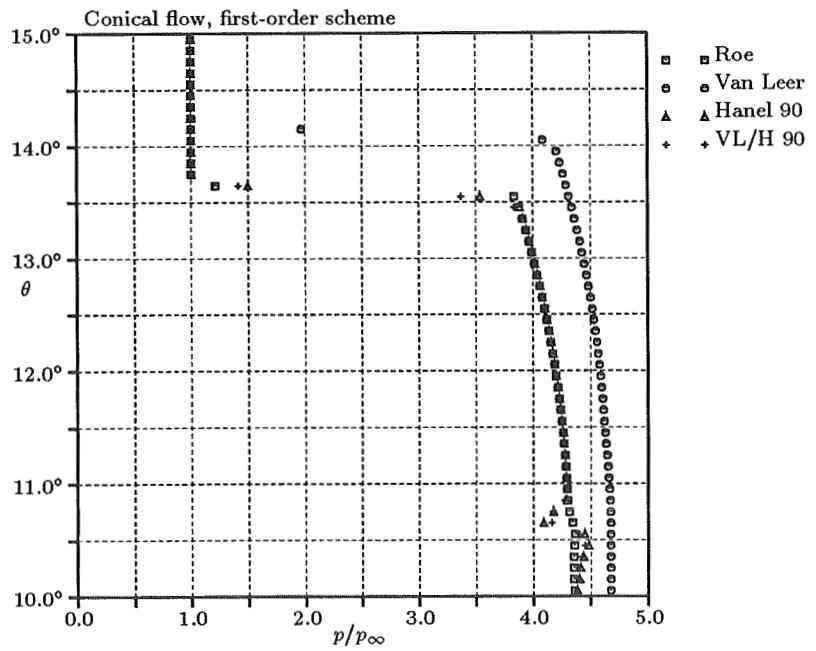


Figure 2: Same solutions as in Figure 1; pressure distributions.

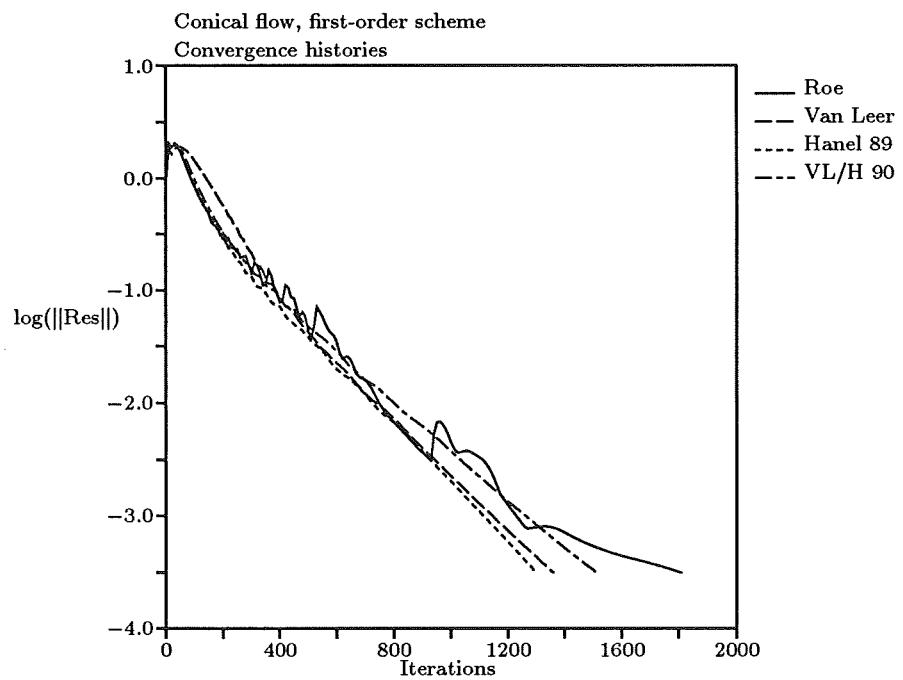


Figure 3: Residual histories of the solutions in Figures 1 and 2.

If indeed this were possible, the form of the flux splitting for small  $v$  would follow immediately from symmetry considerations:

$$F_{\text{mass}}^{\pm} = \frac{1}{2}\rho v + O(v^2), \quad (16)$$

$$F_{\text{mom}\parallel}^{\pm} = \frac{1}{2}p \pm \alpha \rho c v + O(v^2), \quad (17)$$

$$F_{\text{mom}\perp}^{\pm} = \frac{1}{2}\rho u v + O(v^2), \quad (18)$$

$$F_{\text{energy}}^{\pm} = \frac{1}{2}\rho H v + O(v^2), \quad (19)$$

where  $c$  is the sound speed and  $\alpha$  is a positive-valued free parameter. For vanishing  $\alpha$  this splitting leads to central differencing, which will be neutrally stable for  $v = 0$  and unstable for  $v \neq 0$ , if forward time-differencing is used. Positive values of  $\alpha$  will introduce some dissipation, but it is clear that at least a multi-stage time-stepping algorithm must be used to render stability.

A greater problem is that the above flux values for small  $v$  must smoothly join the branches for larger  $v$ , that is, the standard Van Leer-Hänel fluxes. So far, the numerical experience is that the overall flux splitting can not be stabilized. This result, however, is not conclusive; the possibility of constructing a continuously differentiable flux splitting that reduces to the above formulas for slow flow, and is stable for all flow speeds when used in a suitable time-marching scheme, still remains open.

## 7 Multi-dimensional flux-vector splitting

The interpretation of flux-vector splitting as a consequence of positive and negative particle speeds included in a distribution function makes it possible to extend the concept to the multi-dimensional Euler equations in a grid-independent way. For instance, when extending the Beam Scheme into two dimensions, we may consider introducing just one “beam” moving with the average flow velocity, and a circular “front” moving, relative to the average flow, with the dispersion velocity:

$$\phi(q_x, q_y) = \rho_{\text{beam}} \delta \left( \sqrt{(q_x - u)^2 + (q_y - v)^2} \right) + \rho_{\text{front}} \delta \left( \sqrt{(q_x - u)^2 + (q_y - v)^2} - \sigma \right). \quad (20)$$

The standard two-dimensional extension of the distribution function calls for five beams; the two added beams represent the velocity dispersion in the  $y$ -direction. This introduces a directional bias in numerical solutions; in particular, shock waves oblique to the grid may be excessively smeared. The above distribution function should eliminate this effect; the penalty for its use is that the expressions for the fluxes out of a cell are more complicated.

Extension of the Van Leer fluxes in a grid-independent way is more problematic, as these do not derive from a given distribution function. It is possible, *a posteriori*, to find some velocity distribution that explains the one-dimensional Van Leer fluxes; this could then be extended in an omnidirectional way. The resulting fluxes, however, would be somewhat arbitrary, and very complicated. It would be preferable to derive the multi-dimensional version from purely algebraic considerations; this approach remains to be investigated.

## 8 An outlook for flux-vector splitting

Flux-vector splitting is still alive, even for Navier-Stokes applications. It may be mixed with flux-difference splitting, for the sake of preventing numerical diffusion of boundary layers, and there may still be a way to avoid this diffusion in a pure flux-vector splitting. More work needs to be done; the reward could be a simple and robust, easily linearized inviscid-flux formula, with an accuracy rivalling that of Roe's flux formula.

## Acknowledgement

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